

Perturbed Linear-Quadratic Control Problems and Their Probabilistic Representations

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Abstract

We consider some certain nonlinear perturbations of the stochastic linear-quadratic optimization problems and study the connections between their solutions and the corresponding Markovian backward stochastic differential equations (BSDEs). Using the methods of stochastic control, nonlinear partial differential equations (PDEs) and BSDEs, we identify conditions for the solvability of the problem and obtain some regularity properties of the solutions.

Key Words: Stochastic control, HJB equation, FBSDEs,

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1. Introduction

Stochastic optimal control problems and their connections with nonlinear PDEs and backward stochastic differential equations (BSDEs) have been subject of extensive research thanks to their wide range of potential applications in engineering, financial economics and related areas. However, explicit solutions to such problems can be obtained only in a few special cases where the state and control variables usually appear linearly in the state dynamics, for example, the linear-quadratic regulator (LQR) problems. Both theoretical and computational issues arise when some nonlinear terms are added to the system, sometimes representing the effect of a sudden outside force (perturbation) to the variable of interest. In this paper, we consider nonlinear perturbations only in the drift term of a state variable, study the properties of the solution to the corresponding control problem and then compare it with the standard (unperturbed) LQR problems. Our approach involves using the connections between some quasilinear PDEs and a class of forward-backward stochastic differential equations (FBSDEs) of the following Markovian form

$$\begin{aligned} dX(t) &= \mu(t, X(t))dt + \sigma(t, X(t))dW(t), \quad 0 \leq t \leq T \\ dY(t) &= -F(t, X(t), Y(t), Z(t))dt + Z(t)dW(t), \quad 0 \leq t \leq T \\ X(0) &= x; \quad Y(T) = g(X(T)) \end{aligned} \tag{1}$$

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where μ and σ are the *drift* and *diffusion* terms, respectively, of the forward process X ; F is the *driver* term of the backward process Y , and $Y(T) = g(X(T))$ is the terminal condition. We refer the reader to the books by Yong and Zhou (1999), Ma and Yong (1999) and the survey paper by El Karoui et. al. (1997) for the general theory and applications of backward stochastic differential equations (BSDEs). The existence-uniqueness results for nonlinear BSDEs were provided by Pardoux and Peng (1990), Mao (1995), Lepeltier and San Martin (1997, 1998), Kobylanski (2000), Briand and Hu (2006, 2008), and Cetin (2012), among others.

The connections between decoupled FBSDEs and quasilinear PDE's were first stated by Pardoux and Peng (1992), and Peng (1992) by generalising Feynman-Kac representation of PDE's. A version of their results that is relevant to the system (1) is given below:

Theorem 1 (Pardoux and Peng, 1992) *Consider the following parabolic PDE:*

$$\begin{aligned} v_t + \mu v_x + F(t, x, v, \sigma v_x) + \frac{1}{2} \sigma^2 v_{xx} &= 0 \\ v(T, x) &= g(x), \end{aligned} \quad (2)$$

together with the decoupled system of FBSDEs (1). If the PDE (2) has a (classical) solution v , then the pair (Y, Z) with $Y^{s,x}(t) = v(t, X^{s,x}(t))$, and $Z^{s,x}(t) = \sigma(t, X^{s,x}(t))v_x(t, X^{s,x}(t))$ solve the BSDE in (1). Conversely, if the system (1) has a unique (adapted) solution, then $v(t, x) \triangleq Y^{t,x}(t)$ is a viscosity solution to the PDE (2). Moreover, this solution is unique if the coefficients involved are uniformly Lipschitz.

We provide the basic definitions and the notations of the paper below.

1.1 Definitions and Notations

We consider the one-dimensional Euclidean space \mathbb{R} , fixed time-horizon $[0, T]$ and a probability space (Ω, \mathcal{F}, P) where $\mathcal{F} = \{\mathcal{F}_t : 0 \leq t \leq T\}$ is the complete σ -algebra generated by a Brownian motion process W . We define the following function spaces:

- $C^{p,q}([0, T] \times \mathbb{R})$: The space of all real-valued measurable functions $f : [0, T] \times \mathbb{R}$ such that $f(t, x)$ is p (respectively, q) times continuously differentiable with respect to t (respectively, x) where p, q are non-negative integers.
- $L^p_{\mathcal{F}_T}(\Omega)$: The space of \mathcal{F}_T -measurable random variables H such that $E[|H|^p] < \infty$.
- $L^\infty_{\mathcal{F}_T}(\Omega)$: The space of \mathcal{F}_T -measurable essentially bounded random variables.
- $L^p_{\mathcal{F}}([0, T])$: The space of \mathcal{F} -adapted processes f such that $E[\int_0^T |f(t)|^p dt] < \infty$.
- $L^\infty_{\mathcal{F}}([0, T])$: The space of \mathcal{F} -adapted essentially bounded processes.
- $S^p_{\mathcal{F}}(C[0, T])$: The space of \mathcal{F} -adapted continuous processes such that $E[\sup_{0 \leq t \leq T} |f(t)|^p] < \infty$.

For a deterministic function $h(t, x) : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, the subscript notation denotes partial derivatives: $h_t(t, x) = \frac{\partial h}{\partial t}(t, x)$, $h_x(t, x) = \frac{\partial h}{\partial x}(t, x)$ and $h_{xx}(t, x) = \frac{\partial^2 h}{\partial x^2}(t, x)$. In particular, for time dependent functions or ODE's, dot ($\dot{\cdot}$) designates the derivative with respect to time parameter t . The notation $E_t[\cdot]$ will denote the conditional expectation $E[\cdot | \mathcal{F}_t]$. When the initial value of a process X is given at time t , then $E_{t,x}[\cdot]$ refers to $E[\cdot]$ with $X_t = x$. We also use the following facts and notation related to the stochastic control theory. For the details and the proofs of these arguments, one can refer to the books by Fleming and Rishel (1975), Fleming and Soner (2006), or Yong and Zhou (1999).

Let $U \subset \mathbb{R}$ and $x_0 \in \mathbb{R}$. Then for U -valued, \mathcal{F}_t -adapted control processes u_t , consider the following control dependent SDE:

$$\begin{aligned} dX_t &= a(t, X_t, u_t)dt + \sigma(t, X_t)dW_t \\ X_0 &= x_0 \end{aligned} \quad (3)$$

Consider also the cost functional

$$J^u(s, x) = E_{s,x} \left[\int_s^T f(t, X_t, u_t)dt + g(X_T) \right] \quad (4)$$

where the *running cost* F , the *terminal cost* g and the control u belong to some appropriate L_F^p or $L_{F,T}^p$ space. We then define the *value function* as

$$V(s, x) = \inf_{u \in \mathcal{U}^{s,x}} J^u(s, x) \quad (5)$$

where $\mathcal{U}^{s,x}$ is the set of all *admissible controls*, which consists of all $\{F_t, 0 \leq s \leq t \leq T\}$ adapted processes $u = \{u(t), s \leq t \leq T\}$ with values in \mathbb{R} such that $E[\int_0^T |u(t)|^2 dt] < \infty$ and the state process $X^{s,x;u} = X$ in (3) has a unique strong solution in L_F^2 . When there is no ambiguity, the notations $X^{s,x;u}$, $\mathcal{U}([0, T], \mathbb{R})$ will be abbreviated as X and \mathcal{U} , respectively. We use subscript notation X_t and u_t for the state and control processes when the context is clear. If a pair (X^*, u^*) is optimal for the problem (3)-(5) and the value function satisfies $V \in C^{1,2}([0, T] \times \mathbb{R})$, then by dynamic programming principle (DPP) and standard verification theorems, V solves the following second-order nonlinear PDE:

$$0 = \inf_u \{f(t, x, u) + (\mathbf{L}^u v)(t, x)\} \quad (6)$$

with terminal condition $v(T, x) = g(x)$, where \mathbf{L} is the backward evolution operator

$$\mathbf{L}^u v(s, x) = v_s(s, x) + a(s, x, u)v_x(s, x) + \frac{1}{2}\sigma^2 v_{xx}(s, x). \quad (7)$$

The equation (6) is called the *Hamilton-Jacobi-Bellman (HJB) equation* (or *HJB PDE*). The PDE is called *uniformly parabolic* if $\exists c > 0$ such that $|\sigma(t, x)| \geq c$ for all $(t, x) \in [0, T] \times \mathbb{R}$. Such PDEs are known to have unique classical solutions under some regularity and growth conditions, see for example, Fleming and Soner (2006, IV.4) or Yong and Zhou (1999).

Notation 2 We write $u \in \mathcal{U}^{s,x}$ or $u \in \mathcal{U}$ to refer to an admissible control system $(\Omega, F, P, X(\cdot), u(\cdot))$ when the context is clear.

The next section briefly describes how such a system of FBSDEs can be used to study the properties of the solutions to some certain quasilinear HJB PDEs corresponding to the stochastic optimal control problems of the form (3)-(7) where only the drift term of the state process is control-dependent. Such an example is the linear-quadratic regular (LQR) problems where the diffusion term is control-free, and the value function has an explicit (quadratic) form which can be solved analytically or numerically. However, when the drift term of the state equation is not linear, an explicit form of the value function may not be available since the corresponding HJB equation doesn't have an analytic solution, in general. This occurs in the perturbed LQR problems with the drift coefficient having extra nonlinear terms. A class of such nonlinear perturbation were studied by Tsai (1978) without a terminal cost term. See also Nishikawa et al. (1976). For more general cases, some generalized (e.g. *viscosity*) solutions should be considered. Even if a smooth solution exists, there are some other issues to consider: Uniqueness, regularity properties and the numerical computation of the solutions.

2. The PDE and FBSDE Representations

In this section, we first assume that the stochastic control problem (3)-(5) is solvable with the value function $V(t, x) \in C^{1,2}([0, T] \times \mathbb{R})$, and state a general (weak) representation formula for the corresponding FBSDE system, in the spirit of Theorem 1. However, we will consider strong solutions in the rest of the paper.

Lemma 3 Let $V(t, x) \in C^{1,2}([0, T] \times \mathbb{R})$ be a solution to the optimization problem (3)-(5) and $u^* = \arg \min_u \{f(t, x, u) + a(t, x, u)V_x(t, x)\}$ which depends on t and x deterministically through $u^*(t, x) = \pi(t, x, V_x(t, x))$, representing an optimal Markovian control rule $u^*(t, X_t)$. Moreover, assume that for some $p \geq 2$ and for all $s \in [0, T]$, the stochastic integral equation $X_t^{s,x} = x + \int_s^t \sigma(r, X_r) dW_r$ has a weak solution $(\hat{X}, \hat{W}, \hat{F})$ in $L^p_{\hat{F}}([s, T])$ and $f(t, \hat{X}, \pi(t, \hat{X}, V_x(t, \hat{X})) \in L^1_{\hat{F}}([s, T])$. Then
(i) V solves the PDE

$$\begin{aligned} v_t + \frac{1}{2} \sigma^2 v_{xx}(t, x) + \hat{F}(t, x, \sigma v_x(t, x)) &= 0 \\ v(T, x) &= g(x), \end{aligned}$$

where $\hat{F}(t, x, z) = a(t, x, \pi(t, x, \sigma^{-1}(t, x)z)) + f(t, x, \pi(t, x, \sigma^{-1}(t, x)z))$.

(ii) A solution to the system

$$Y_t^{s,x} = g(\hat{X}_T) + \int_t^T \hat{F}(r, \hat{X}_r, Z_r) dr - \int_t^T Z_r d\hat{W}_r \quad (8)$$

is given by $Y_t^{s,x} = V(t, \hat{X}_t^{s,x})$, $Z_t^{s,x} = \sigma V_x(t, \hat{X}_t^{s,x})$.

Proof. The part (i) follows from the stochastic optimal control and DPP arguments in the previous section. For part (ii), let the operator \mathbb{L}^u be as in (7) corresponding to the SDE (3). Then applying Ito's rule to $Y(t) = V(t, \hat{X}(t))$ and using (6) with $u^* = u^*(t, \hat{X}) = \pi(t, \hat{X}, \sigma V_x(t, \hat{X}))$, we get:

$$\begin{aligned} dY &= \{\mathbb{L}^{u^*} V(t, \hat{X}) - a(t, \hat{X}, \pi(t, \hat{X}, V_x(t, \hat{X})))\}dt + \sigma(t, \hat{X}) V_x(t, \hat{X}) d\hat{W} \\ &= -\{f(t, \hat{X}, \pi(t, \hat{X}, V_x(t, \hat{X}))) + a(t, \hat{X}, \pi(t, \hat{X}, V_x(t, \hat{X})))\}dt + Z d\hat{W} \\ &= -\hat{F}(t, \hat{X}, Z)dt + Z d\hat{W}, \end{aligned}$$

which also corresponds to the BSDE representation in (1) and (2), with $\mu = 0$.¹ ■

Under some standard regularity and growth conditions (Lipshitz parameters, linear growth etc.), the HJB PDEs (and the corresponding BSDEs) have unique solutions. However, the equations that we consider are quite general and there is no guarantee that a (classical) solution exists or if it exists whether the solution is unique (in a suitable space). In the next subsection, our particular interest will be on a more specific modeling of the state variable X and the cost functional f , which has some interesting applications in economics (controlling macroeconomic variables) and in engineering (target tracking). We keep the model coefficients as general as possible in this section to motivate the choice of the particular functions used in the next section.

2.1 The Additively Separable Drift and Cost Functions

We consider the following controlled state process $X = X^u$, for $0 < t \leq T$:

$$\begin{aligned} dX_t &= (\mu(t, X_t) + B(t, X_t)u_t)dt + \sigma(t, X_t)dW(t), \\ X_0 &= x_0. \end{aligned} \tag{9}$$

where the deterministic functions $B(t, x)$, $\mu(t, x)$ and $\sigma(t, x)$ are continuous functions of their arguments. Moreover, for $(t, x, u) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ given, we introduce the following running cost function in (4):

$$f(t, x, u) = h(t, x) + k(t, x)u^2 \tag{10}$$

where $k(t, x)$ is a positive continuous function on $[0, T] \times \mathbb{R}$, and the terminal cost $g(x)$ is such that $g(X_T) \in L^1_{F_T}$.

Again, by a heuristic application of DPP as in (6), the value function $V(s, x)$ in equation (5) satisfies the HJB equation

$$\begin{aligned} 0 &= \inf_u \{f(t, x, u) + (\mathbb{L}^u V)(t, x)\} \\ &= V_t + \frac{1}{2}\sigma^2 V_{xx}(t, x) + h(t, x) + \mu V_x(t, x) \\ &\quad + \inf_u \{k(t, x)u^2 + B(t, x)u V_x\}. \end{aligned} \tag{11}$$

¹Note that the Brownian motion process W in equation (8) is not necessarily the original one due to the weak representation approach.

Clearly, the minimum of $k(t, x)u^2 + B(t, x)uV_x$ in (11) is $\frac{-B^2(t, x)}{4k(t, x)}(V_x)^2$ with the minimizer

$$u^* = \pi(t, x, V_x(t, x)) = \frac{-B(t, x)}{2k(t, x)}V_x. \quad (12)$$

Plugging (12) into the equation (11), the HJB PDE takes the form of

$$\begin{aligned} v_t + \frac{1}{2}\sigma^2 v_{xx}(t, x) + h(t, x) + \mu v_x(t, x) - \frac{B^2(t, x)}{4k(t, x)}(v_x)^2 &= 0 \\ v(T, x) &= g(x) \end{aligned} \quad (13)$$

which can also be written as

$$\begin{aligned} v_t + \frac{1}{2}\sigma^2 v_{xx}(t, x) + \mu v_x(t, x) + F(t, x, \sigma v_x(t, x)) &= 0 \\ v(T, x) &= g(x) \end{aligned} \quad (14)$$

with

$$F(t, x, z) = h(t, x) - \frac{H(t, x)}{2}z^2, \text{ for } (t, x, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}, \quad (15)$$

where $H(t, x) = \frac{B^2(t, x)}{2k(t, x)\sigma^2(t, x)}$.

Remark 4 (a) In a standard LQR problem, the state dynamics in (9) has the usual linear drift and diffusion coefficients, $k(t, x) = k(t)$ and $B(t, x) = B(t)$. Moreover, $h(t, x)$ and $g(x)$ are quadratic cost functions. It is well known that the condition $k(\cdot) \geq 0$ ($k(\cdot) > 0$, respectively) is a necessary (sufficient, respectively) condition for the standard LQR problem to be solvable.

(b) If the value function solves the quasilinear PDE (13), then the PDE representation given by (14)-(15) can be utilized for a BSDE representation of the problem by Theorem 1 or Lemma 3.

2.2 The FBSDE Interpretation

In view of Theorem 1 (and using the similar steps as in Lemma 3), if the value function $V(t, x)$ is a smooth solution to the equations (14)-(15), then the pair $(Y_t^{s, x}, Z_t^{s, x})$ with $Y_t = V(t, \tilde{X}_t)$ and $Z_t = \sigma(t, \tilde{X}_t)V_x(t, \tilde{X}_t)$ is a solution to the BSDE

$$\begin{aligned} dY_t^{s, x} &= -F(t, \tilde{X}_t, Z_t)dt + Z_t dW_t \\ Y_T^{s, x} &= g(\tilde{X}_T) \end{aligned} \quad (16)$$

with

$$\tilde{X}_t^{s, x} = x + \int_s^t \mu(r, \tilde{X}_r^{s, x})dr + \int_s^t \sigma(r, \tilde{X}_r^{s, x})dW_r, \quad (17)$$

and $F(t, x, z)$ as in (15). However, in general, we don't know if the PDE has a classical solution since the function $F(s, x, z)$ doesn't satisfy the usual Lipschitz or linear growth conditions in the state variables x and z . It also gets more complicated when the terminal condition $g(X_T)$ is not bounded.

Remark 5 (a) The representation of (16)-(17) as a FBSDE system is not unique. Another representation (as in Lemma 3) may be given by the following system, by eliminating the drift term of the forward process:

$$\begin{aligned}\hat{X}_t^{s,x} &= x + \int_s^t \sigma(r, \hat{X}_r) dW_r \\ Y_t^{s,x} &= g(\hat{X}_T) + \int_t^T \hat{F}(r, \hat{X}_r, Z_r) dr - \int_t^T Z_r dW_r\end{aligned}\tag{18}$$

where the new driver function is $\hat{F}(t, x, z) = F(t, x, z) + \frac{\mu(t, x)}{\sigma(t, x)}z$. Each representation has some advantages depending on the complexity level of the forward and backward equations in (16)-(18). In this section, the representation (16) will be used frequently based on the assumption that the forward state dynamics (17) has a unique solution.

(b) When the diffusion term σ is time dependent only, the uniqueness of the solutions to (16) can be shown even for more general drift terms, as in Cetin (2012a). For the systems with state-dependent diffusion terms $\sigma(t, x)$ and nonlinear drift terms, some more regularity or monotonicity assumptions would be needed.

When the expressions $\sigma(t, x)$ and $H(t, x)$ are time-dependent only, we have the following Theorem which is a special case of a result from Cetin (2012) where the driver F is also allowed to depend on y :

Theorem 6 Assume that

- (i) the SDE (17) has a unique solution \tilde{X} in $L_F^2[0, T]$ with a.s. continuous paths
- (ii) the value function $V(t, x)$ in (5) is a smooth solution of (14)-(15)
- (iii) $\sigma(t, x) = \sigma(t)$, satisfying $|\sigma(\cdot)| > \delta > 0$ uniformly on $[0, T]$.
- (iv) the function $H(t, x) = H(t)$ is differentiable, and is such that $\left| \dot{H}(\cdot)/H(\cdot) \right|$ is bounded on $[0, T]$.

Then the BSDE (16) has a unique solution (Y, Z) in $L_{F_T}^2 \times L_F^2$.

Remark 7 Since the term $h(s, \tilde{X}(s))$ is not bounded, the existence of a solution is not guaranteed in general. However, if a function $v(t, x)$ satisfies the HJB PDE (14)-(15), then thanks to the monotonic transformations $U_t = \exp(-H(t).Y)$ and $\Lambda_t = -H(t)U_t Z_t$, the pair $[\exp(-H(t).v(t, \tilde{X}_t)), -H(t)\sigma(t)v_x(t, \tilde{X}_t)\exp(-H(t).v(t, \tilde{X}_t))]$ solves the BSDE with $0 < U_t = \exp(-H(t).v(t, \tilde{X}_t)) < 1$.

Corollary 8 If the value function $V(t, x)$ satisfies the HJB PDE (14)-(15), then it is the unique solution of (14)-(15).

2.3 A Relevant LQR Problem with State-Independent Diffusion

Now consider the linear state dynamics $\mu(t, x) = A(t)x$ and $\sigma(t, x) = \sigma(t)$ in addition to the quadratic cost functions $f(t, x, u) = e^{-\lambda t}[(x - \xi(t))^2 + k_1(t)u^2]$ and $g(x) = k_2(x - \xi(T))^2$,

where all the time-dependent functions are continuous. When $\xi(\cdot) = 0$, the problem reduces to the standard LQR problem. By an appeal to the standard verification theorems for the stochastic control problems (Fleming and Soner, 2006), the system (14)-(15) can be shown to have a unique smooth solution $v(t, x)$, to the corresponding LQR optimization problem in the following form:

$$\begin{aligned} V(t, x) &= P(t)x^2 + K(t)x + N(t) \\ V(T, x) &= k_2(x - \xi(T))^2, \end{aligned} \quad (19)$$

where, for $s \leq t < T$, the functions P, K and N solves the system of the ODEs below:

$$\dot{P}(t) + e^{-\lambda t} + 2AP(t) - e^{\lambda t} \frac{B^2}{k_1} P^2(t) = 0, \quad P(T) = k_2 \geq 0, \quad (20)$$

$$\dot{K}(t) + (A(t) - e^{\lambda t} \frac{B^2}{k_1} P(t))K(t) - 2e^{-\lambda t} \xi(t) = 0, \quad K(T) = -2e^{-\lambda T} \xi(T), \quad (21)$$

$$\dot{N}(t) + \sigma^2 P(t) - \frac{e^{\lambda t} B^2}{4 k_1} K^2(t) + e^{-\lambda t} \xi^2(t) = 0, \quad N(T) = e^{-\lambda T} \xi^2(T). \quad (22)$$

using the notation $\dot{f}(t) \equiv \frac{df(t)}{dt}$ for time derivatives. It is well known (e.g., Fleming and Rishel, p.89) that the Riccati ODE (20) has a non-negative (positive if $k_2 > 0$) solution $P(\cdot) \in C^1[0, T]$. Consequently, the linear first-order equations (21) and (22) also have unique solutions in $C^1[0, T]$. Moreover, an optimal control process is given by

$$u_t^* = -e^{\lambda t} \frac{B}{2k_1}(t)[2P(t)X_t^* + K(t)]$$

and the optimized state process which is given by the SDE

$$\begin{aligned} dX_t^* &= (A(t)X_t^* + B(t)u(t, X_t^*))dt + \sigma(t)dW_t, \\ &= \{(A(t) - \frac{e^{\lambda t} B}{2 k_1} K(t) - e^{\lambda t} \frac{B^2 P}{k_1}(t))X_t^*\}dt + \sigma(t)dW_t \end{aligned} \quad (23)$$

is a Gaussian process on $[0, T]$.

The case with $\sigma(t, x) = \sigma(t)x$ is also similar: The solution is a quadratic function of x with time dependent parameters being solutions to ODEs similar to those above but we are not going to provide the details here (note that the Theorem 6 doesn't apply directly in this case). In both cases, the value function can be obtained by solving the corresponding ODEs numerically. This can be done efficiently even in high dimensions so it is not necessary to consider a FBSDE approach to solve such problems.² However when there is no explicit solution available, then the FBSDE approach could be preferable, especially when the corresponding control problem or when the PDE involves a state variable in higher dimensions. We describe such a nonlinear application which is a generalized version of an example from Tsai (1978).

²Actually, it is shown in Cetin (2006) that solving a corresponding FBSDE system will result in a sequence of iterations which are equivalent to solving the related ODEs above using Euler discretization.

3. Nonlinear State Dynamics with State-Independent Diffusion

Now consider a non-linear drift term while diffusion coefficient is still depending only on time:

$$\begin{aligned} dX_t &= [A(t)X_t + \delta r(t, X_t) + B(t)u_t]dt + \sigma(t)dW(t), \\ X_0 &= x_0 > 0 \end{aligned} \quad (24)$$

where $\sigma(\cdot)$ is bounded away from 0 on the interval $[0, T]$, $p(t, x)$ is a non-linear perturbation term, and δ is the perturbation constant. The system reduces to a linear one when $\delta = 0$. Let the cost functional for this perturbed problem be given

$$J^u(s, x) = E_{s,x} \int_s^T [l(t)(X_t - \xi(t))^2 + k(t)u_t^2]dt \quad (25)$$

where $k(\cdot) > 0$ and $\xi(\cdot)$ is continuous on $[0, T]$. Define the value function as $V(s, x) = \inf_{u \in \mathcal{U}} J^u(s, x)$, and let the control set \mathcal{U} consist of all square integrable adapted processes u_t such that the equation (24) has a unique solution in $L^2([s, T], \mathbb{R})$ (however it is sufficient to consider only the feedback controls of the Markovian form). This quadratic optimization problem cannot be solved explicitly unless $\delta = 0$ however assuming that the SDE (24) has a solution for a sufficiently rich set of the control processes, including the candidate optimal control $u^* = \frac{-B(t)}{2k(t)}V_x$ (from (12)) and $u = 0$, and the optimization problem (4) is solvable, we can identify a corresponding FBSDE system to characterize the solution. Since the terminal condition is bounded (in this case zero, for simplicity), using the methods of the parabolic PDEs and stochastic analysis, it can be shown to have a smooth solution (as in Tsai, 1978). For more general state equations or terminal conditions, one can only expect to get a less smooth (viscosity) solution, using either methods of the PDEs (as in Fleming and Soner, 2006) or those of the FBSDEs. We now state some assumptions that will be used for the main results of this section:

Condition 9 Consider the equations (24)-(25) and let $\mu(t, x) = A(t)x + \delta r(t, x)$, $\bar{\mu}(t, x, u) = \mu(t, x) + B(t)u$.

- (i) The time dependent functions A, B, k, l and σ are continuous on $[0, T]$.
- (ii) The function $l(t)$ is nonnegative, $k(t)$ is positive on $[0, T]$ and is bounded away from zero: For some $\epsilon > 0$, $k(\cdot) > \epsilon$.
- (iii) The function $H(t) = \frac{B^2(t)}{k(t)\sigma^2(t)}$ is continuously differentiable in $(0, T)$ and is bounded away from zero.
- (iv) (Monotonicity condition) For all $t \in [0, T]$ and $x \in \mathbb{R}$, $xr(t, x) \leq C(1 + x^2)$, for some constant $C > 0$.
- (v) (Monotonicity condition): For all $t \in [0, T]$ and $x, y \in \mathbb{R}$, $(x - y)r(t, x) - r(t, y) \leq K(x - y)^2$, for some constant $K > 0$.

Lemma 10 Assume that the functions A, B and σ satisfy the Condition 9 (i), $r(t, x)$ is locally Lipschitz and satisfies the Condition 9 (iv) above. Then,
(a) For every initial condition $\tilde{X}_s = x$, the control-free state equation

$$\tilde{X}_t^{s,x} = x + \int_s^t [A(v)\tilde{X}_v^{s,x} + \delta r(v, \tilde{X}_v^{s,x})] dv + \int_s^t \sigma(v) dW_v \quad (26)$$

has a unique strong solution $\tilde{X}_t^{s,x}$ in $L^p([s, T], \mathbb{R})$, for all $p \geq 2$. Moreover, with $\tilde{C} = \max\{\delta C + \frac{p-1}{2} \max_{s \leq t \leq T} \sigma^2(t), \delta C + \max_{s \leq t \leq T} A(t)\}$, it satisfies the following moment estimate for every $t \in [s, T]$:

$$E_{s,x} \left| \tilde{X}_t^{s,x} \right|^p \leq 2^{\frac{p}{2}-1} (1 + |x|^p) e^{\tilde{C}p(t-s)}.$$

(b) Let $u_t = u(t, X_t)$ be a Markovian feedback control where $u(t, x)$ is locally Lipschitz with respect to x and also satisfies Condition 9 (iv), for some generic constant $C > 0$. Then the SDE (24) has a unique solution in $L^p([s, T], \mathbb{R})$, for all $p \geq 2$, too, and a similar moment estimate as in part (a) holds.

(c) If both $r(t, x)$ and $u(t, x)$ satisfy a linear growth condition in x , then the unique solution of (24) is in $S^p([s, T], \mathbb{R})$ with the estimate

$$E_{s,x} \sup_{s \leq t \leq T} |X_t^{s,x}|^p \leq (1 + |x|^p) e^{C_1(T-s)},$$

where C_1 is a constant depending on p, T, s and the linear growth factor.

Proof. (a) By (iv), for all $t \in [s, T]$ and $x \in \mathbb{R}$, $x\mu(t, x) + \frac{p-1}{2}\sigma^2(t) \leq (A(t) + \delta C)x^2 + \delta C + \frac{p-1}{2}\sigma^2(t) \leq \tilde{C}(1 + x^2)$. Since $\mu(t, x)$ is also locally Lipschitz, the result follows from the standard estimates based on the Lyapunov function approach. See, for example, Mao (1997), Ch. 3, Theorem 4.1. The proof of part (b) is similar, replacing $\mu(t, x)$ with $\bar{\mu}(t, x, u(t, x))$, and part (c) is a standard result for the SDEs with coefficients of linear growth.

■

Lemma 11 Let the assumptions of Condition 9 hold and u^* be an optimal control rule: $u_t^* = u(t, X_t^*)$ with $V(s, x) = J^{u^*}(s, x)$ and $X_t^* = X_t^{u^*}$. Moreover, let $X_s^* = x = \tilde{X}_s$. Then we have the following estimates:

(a) There is a positive constant C such that $0 \leq V(s, x) \leq C(1 + x^2)$, for every $x \in \mathbb{R}$, uniformly on $[0, T]$.

(b) $E_{s,x} \int_s^T |u_t^*|^2 dt \leq \frac{C}{\epsilon} (1 + |x|^2)$ where C is as in part (a)

(c) $E_{s,x} (X_t^* - \tilde{X}_t)^2 \leq \tilde{C}(1 + |x|^2)$, for some positive constant \tilde{C} .

Proof. (a) For $u = 0$, since $l(\cdot) \geq 0$, we get $V(s, x) \leq J^0(s, x) \leq E_{0,x} \int_0^T [l(t)(\tilde{X}_t - \xi(t))^2] dt$. Since $\xi(\cdot)$ and $l(\cdot)$ are bounded on $[0, T]$, the result follows from Lemma 10 with $p = 2$.

(b) By (25) and assumption (ii) of Condition 9, $V(s, x) = E_{s,x} \int_s^T [l(t)(X_t^* - \xi(t))^2 + k(t)u_t^{*2}] dt \geq \epsilon E_{s,x} \int_s^T |u_t^*|^2 dt$. Hence the inequality is a result of part (a).

(c) First note that $d(X_t^* - \tilde{X}_t) = \{A(t)(X_t^* - \tilde{X}_t) + \delta(r(t, X_t^*) - r(t, \tilde{X}_t)) + B(t)u_t^*\}dt$ and let $\Delta(t) = X_t^* - \tilde{X}_t$, for $s \leq t \leq T$. Then by Ito's rule,

$$\Delta^2(t) = \int_s^t 2A(v)\Delta^2(v)dv + \int_s^t 2\delta(r(t, X_t^*) - r(t, \tilde{X}_t))\Delta(v)dv + \int_s^t 2B(v)u_v^*\Delta(v)dv.$$

By applying the monotonicity assumption (v) of Condition 9 to the second integral and the inequality $2ab \leq a^2 + b^2$ to the third one above, we get

$$0 \leq \Delta^2(t) \leq \int_s^t \{(2A(v) + 2\delta K + 1)\Delta^2(v)dv + \int_s^t (B(v)u_v^*)^2 dv,$$

where K is as in Condition 9 (v). Since $B(\cdot)$ is bounded on $[0, T]$, by Lemma 11, $E_{s,x} \int_s^t |B(v)u_v^*|^2 dv \leq C_1(1 + |x|^2)$ for some positive constant C_1 . Moreover, $\exists M > 0$ such that $2A(t) + 2\delta K + 1 \leq M$, for $0 \leq t \leq T$. Therefore, by letting $g(s, x; t) = g(t) = E_{s,x}\Delta^2(t)$, we obtain the inequality $0 \leq g(t) \leq C_1(1 + |x|^2) + M \int_s^t g(v)dv$. By Gronwall's inequality, $g(t) \leq C_1(1 + |x|^2)(1 + M \int_s^t e^{M(t-v)} dv)$ and hence the result follows. ■

Now, by an appeal to Lemma 10 part (a) with $p = 2$, and part (c) of the Lemma above, we get the following Corollary:

Corollary 12 (a) $E_{s,x}|X_t^*|^2 \leq K_1(1 + |x|^2)$, for some positive constant K_1 .
(b) $E_{s,x}|X_t^*| \leq K_2(1 + |x|)$, for some positive constant K_2 .

Theorem 13 Let the assumptions (i)-(v) hold and consider the perturbed state dynamics (24) together with the cost function (25) and the value function $V(s, x)$. Then,

(a) The value function $V(s, x)$ is in $C^{1,2}([0, T] \times \mathbb{R})$ such that

(i) it satisfies the HJB equation

$$\begin{aligned} v_t(t, x) + \frac{1}{2}\sigma^2(t)v_{xx}(t, x) + F(t, x, \sigma(t)v_x) + \delta r(t, x)v_x(t, x) &= 0 \\ v(T, x) &= 0, \end{aligned} \quad (27)$$

where $F(t, x, z) = (x - \xi(t))^2 - \frac{H(t)z^2}{2}$ over $[0, T] \times \mathbb{R}$.

(ii) $\exists C > 0$ such that $|V_x(s, x)| \leq C(1 + |x|)$, uniformly for $s \in [0, T]$.

(b) The process $u_t^* = u(t, X_t) = \frac{-B(t)}{2k(t)}V_x(t, X_t)$ is an admissible feedback control rule such that $u(s, x)$ is the unique minimizer of $J^u(s, x)$, for all $(s, x, u) \in [0, T] \times \mathbb{R} \times \mathbb{R}$. Moreover, $u(\cdot, x)$ is a locally Lipschitz function of x , and satisfies a linear growth condition in x (as in part (a)(ii) above).

Proof. (a) The proof relies on the approximation of the domain $[0, T] \times \mathbb{R}$ by the compact subsets, and the standard but lengthy localization arguments and passages to the limit for the Cauchy problems of second order parabolic equations, as in Tsai (1978) and Fleming and Rishel (1975). We skip these technical details since the steps involved are very similar to those of Lemma 3.1 and Lemma 3.2 of Tsai (1978).

(b) From the more general equations (11) and (12), it is easy to see that $u(s, x) = \frac{-B(t)}{2k(t)}V_x(t, x)$ is the unique minimizer of $J^u(s, x)$ in (25). It has a linear growth in x by part (a)(ii), and is square integrable by Lemma 11. It is locally Lipschitz in x uniformly in t since $\frac{B(t)}{2k(t)}$ is bounded on $[0, T]$, and $V_x(t, x)$ is differentiable with respect to x .

Remark 14 *Since no specific growth condition is assumed on the perturbation term $r(t, x)$, except the monotonicity and local Lipschitz properties, the standard verification theorems and dynamic programming principle of stochastic control theory may not directly apply. Again, some localization techniques would help to ensure the uniqueness of the solutions to the PDE (27).³ However we follow a BSDE approach to prove the uniqueness. We then provide a probabilistic representation of both the value function and the optimal control.*

■

Theorem 15 *Let $\tilde{X}_t^{s,x}$, X_t^* , u_t^* and $V(t, x)$ be as before, and introduce the pair $(Y_t^{s,x}, Z_t^{s,x})$ as $Y_t = V(t, \tilde{X}_t)$ and $Z_t = \sigma(t)V_x(t, \tilde{X}_t)$. Then,*

(i) *The pair $(Y_t^{s,x}, Z_t^{s,x})$ is the unique solution to the BSDE*

$$dY_t^{s,x} = -F(t, \tilde{X}_t, Z_t)dt + Z_t dW_t, \quad Y_T^{s,x} = 0. \quad (28)$$

Moreover, $V(t, x) = Y_t^{t,x}$ (the value of Y_t when $X_t = x$), and $u(t, x) = \frac{-B}{2k\sigma}(t)Z_t^{t,x}$ for $x > 0$ and $t \in [0, T]$. An optimal (feedback) control for the problem is given .

(ii) *The value function is the unique classical solution of the PDE (27) with a quadratic growth in x . Moreover, the control process u_t^* is the unique optimal control for the optimization problem (24)-(25).*

Proof. The uniqueness follows from the Theorem 6 and the Corollary 8. Then the optimality is a direct result of Part (ii) is a result of Theorem 13 and the uniqueness. The BSDE representation in (i) is obtained similar to that of (16)-(17) and Remark 5 by following the same steps as in Lemma 3. ■

Corollary 16 *The triple $(\tilde{X}_t^{s,x}, Y_t^{s,x}, Z_t^{s,x})$ satisfies*

$$(a) \sup_{s \leq t \leq T} E_{s,x} |Y_t| + E_{s,x} \left[\int_s^T Z_t^2 dt + \int_s^T \tilde{X}_t^2 dt \right] \leq C(1 + x^2).$$

$$(b) \text{ If the perturbation term } r(t, x) \text{ has a linear growth, then } E_{s,x} \sup_{s \leq t \leq T} |Y_t| + E_{s,x} \left[\int_s^T Z_t^2 dt + \int_s^T \tilde{X}_t^2 dt \right] \leq C(1 + x^2) \text{ also holds.}$$

³Tsai (1978) take advantage of the differentiability (and the existence of an upper bound on the derivative) that doesn't apply here.

Proof. It is a result of Lemma 11, Theorem 13 part (a) (ii), and Theorem 15, by utilizing Lemma 10 (a) for the proof of part (a), and Lemma 10 (c) for the proof of part (b), with $p = 2$. ■

The following example is a slightly generalization of an application from Tsai (1978). The well-posedness of more general cases are discussed in Cetin (2012a).

Example 17 Consider the following controlled state dynamics with constant parameters:

$$\begin{aligned} dX_t &= (-\delta X_t^3 + Bu_t)dt - \sigma dW_t \\ X_0 &= x_0 > 0. \end{aligned}$$

Let the cost functional for this perturbed problem be given by $J^{\delta,u}(s, x) = E_{s,x} \int_s^1 [(X_t - \xi)^2 + ku_t^2]dt$ where $k > 0$ and the value function is $V^\delta(s, x) = \inf_u J^{\delta,u}(s, x)$. By the Theorem above, $V^\delta(s, x)$ is the unique solution to the PDE

$$\begin{aligned} v_t(t, x) + \frac{1}{2}\sigma^2 v_{xx}(t, x) + (x - \xi)^2 - \delta x^3 v_x(t, x) - Cv_x^2(t, x)/4 &= 0 \\ v(T, x) &= 0 \end{aligned}$$

where $C = B^2/k > 0$ over $[0, T]$. Again, it is not likely to obtain an explicit solution of this nonlinear and one needs to follow a numerical procedure to solve the equation and hence describe the behavior of the optimal action (control). Note that for the unperturbed LQR problem ($\delta = 0$) with the constant parameters C and ξ , it can be verified (using the results in subsection 2.3) that the solution is given by the quadratic expression $V^0(s, x) = \lambda(t)(x - \xi)^2 + \gamma(t)$, where $\lambda(t) = \frac{\tanh(\sqrt{C}(1-t))}{\sqrt{C}}$ and $\gamma(t) = \frac{\sigma^2}{C} \ln \cosh(\sqrt{C}(1-t))$. Moreover, the optimal control $u_t^{*,0}$ is a linear feedback control: $u_t^{*,0} = -\frac{B}{2k}(t)V_x^0(t, x) = \frac{-\text{sign}(B)}{\sqrt{k}} \tanh(\frac{|B|}{\sqrt{k}}(1-t))(x - \xi)$. When the state equation deviates from a linear dynamics significantly, then it may be important to know how the corresponding optimal action and the value function differ from the unperturbed LQR setup. The discussion of this problem and its numerical solution using a probabilistic approach are considered in Cetin (2012b). One can also refer to Tsai (1978) and Nishikawa et al. (1976) for a PDE approach for the details. Some results are given below without proof:

- The value function $V^\delta(s, x)$ can be approximated as follows:

$$V^\delta(s, x) = V^0(s, x) + 2\delta[K_1(s)x^4 + K_2(s)x^2] + O(\delta^2)$$

where $V^0(s, x)$ is the value function for the unperturbed problem and $K_1(\cdot), K_2(\cdot) \in C^1[0, 1]$.

- The optimal control is

$$u^{*,\delta}(s, x) = u^{*,0}(s, x) - 2\delta[4K_1(s)x^3 + 2K_2(s)x] + O(\delta^2)$$

where $u^{*,0}(s, x)$ is as above.

- $u^{*,\delta}(s, x) \rightarrow u^{*,0}(s, x)$, as $\delta \rightarrow 0$ uniformly on $[0, T] \times Q$, for any compact set Q of \mathbb{R} .

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